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Random Differential Inclusions in Banach Spaces

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1. INTRODUCTION

There are many problems in applied mathematics that lead us to the study of dynamical systems having velocities not uniquely determined by the state of the system, but depending only loosely upon it. In these cases the classical equation $\dot{x}(t)=f(t, x(t))$ describing the dynamics of the system is replaced by an equation of the form $\dot{x}(t) \in F(t, x(t))$, where $F(\cdot, \cdot)$ is a set valued map (multifunction). The initial motivation for the study of differential inclusions came from control theory. There we encounter dynamical systems described by an equation of the form $\dot{x}(t)=f(t, x(t), u(t))$ where $u(t)$ is the control parameter. Every solution of this equation also solves the differential inclusion $\dot{x}(t) \in F(t, x(t))$ where $F(t, x(t))=\bigcup_{u \in U(t)} f(t, x(t), u)$ and this formulation has the advantage that the control variables do not appear explicitly. Furthermore in an optimal control problem, with control region constant, the Pontryagin maximum principal may be put in the form of a two-point boundary value problem for a differential inclusion with upper semicontinuous right-hand side. Also implicit differential equations $f(t, x(t), \dot{x}(t))=0$ can be viewed as differential inclusions, where the multifunction $F(\cdot, \cdot)$ is defined by $F(t, x)=\{z \in X: f(t, x, z)=0\}$. In addition differential inclusions appear naturally in the study of nonsmooth Hamiltonian systems and nonsmooth optimal control problems (see Clarke [6]), in optimization theory (see Cesari [5]), and in differential equations $\dot{x}(t)=f(t, x(t))$, $x(0)=x_0$, with a discontinuous right-hand side (see Filippov [14]). A detailed study of differential inclusions in $\mathbb{R}^n$ can be found in the recent book by Aubin and Cellina [1].

The purpose of this note is to prove existence results for large classes of random differential inclusions. In doing that we also obtain some interesting deterministic results. The organization of the paper is as follows. In the next section we present some of the basic mathematical background that is needed to follow this work. In Section 3 we prove some auxiliary results which we are going to need later and which are also interesting on

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their own as general results about multifunctions. In Section 4 we prove two deterministic existence results, and finally in Section 5 we use them to obtain existence results for random differential inclusions.

2. Preliminaries

Let (Ω, Σ, μ) be a complete σ-finite measure space and X a separable Banach space, with X^* being its topological dual. We will use the notations

$$P_{f(c)}(X) = \{A \subseteq X\colon \text{nonempty, closed, (convex)}\}$$

$$P_{(w)k(c)}(X) = \{A \subseteq X\colon \text{nonempty, (w)-compact, (convex)}\}.$$

For $A \in 2^X \setminus \{\phi\}$ we set $|A| = \sup_{x \in A} \|x\|$ and by $\sigma(\cdot)_A$ we denote its support function, i.e., for all $x^* \in X^*$, $\sigma(x^*)_A = \sup_{x \in A} (x^*, x)$.

A multifunction (set valued function) $F\colon \Omega \to P_f(X)$ is said to be measurable if it satisfies any of the following equivalent conditions:

(i) $\omega \to d(x)_{F(\omega)} = \inf_{z \in F(\omega)} \|x - z\|$ is measurable for all $x \in X$,

(ii) there exists a sequence $\{f_n(\cdot)\}_{n \geqslant 1}$ of measurable selectors of $F(\cdot)$ s.t. $F(\omega) = \text{cl}\{f_n(\omega)\}_{n \geqslant 1}$ for all $\omega \in \Omega$ (Castaing's representation),

(iii) $\text{Gr}\, F \in \{(\omega, x) \in \Omega \times X\colon x \in F(\omega)\} \in \Sigma \times B(X)$ where $B(X)$ is the Borel σ-field of X.

For a detailed treatment of measurable multifunctions we refer to any of the excellent references Castaing and Valadier [4], Himmelberg [17], Rockafellar [28], and Wagner [30].

We denote by S_F^1 the set of all selectors of $F(\cdot)$ that belong to the Lebesgue–Bochner space $L_X^1(\Omega)$, i.e.,

$$S_F^1 = \{f(\cdot) \in L_X^1(\Omega)\colon f(\omega) \in F(\omega)\ \mu - \text{a.e.}\}.$$

It is easy to see that this is closed and it is nonempty if and only if $\inf_{x \in F(\omega)} \|x\| \in L_+^1(\Omega)$. Using this set we can define an integral for multifunctions. This integral was first introduced by Aumann [2] for $X = \mathbb{R}^n$ as the natural generalization of the Minkowski sum of sets and of the integral of single valued functions. So we have

$$\int_\Omega F(\omega)\, d\mu(\omega) = \left\{\int_\Omega f(\omega)\, d\mu(\omega)\colon f(\cdot) \in S_F^1\right\}$$

where the vector valued integral is in the sense of Bochner. We say that $F(\cdot)$ is integrably bounded if and only if $F(\cdot)$ is measurable and $|F(\cdot)| \in L_+^1(\Omega)$. Also recall that a multifunction $F\colon \Omega \times X \to P_f(X)$ is said to be separable if there exists a countable set $D \subseteq X$ and a negligible set $N \in \Sigma$, $\mu(N)=0$ s.t. $\overline{F(\omega, K)} = \overline{F(\omega, K \cap D)}$ for all $\omega \in \Omega \backslash N$ and all $K \subseteq Y$ closed. Clearly if $F(\cdot, \cdot)$ is separable, then for all $z \in X$ and all $x^* \in X^*$, $d(z)_{F(\cdot,\cdot)}$ and $\sigma(x^*)_{F(\cdot,\cdot)}$ are separable random processes.

Next we will recall some continuity concepts associated with multifunctions. So let Y be a locally convex space and $F: X \to 2^Y \backslash \{\phi\}$. We say that $F(\cdot)$ is u.s.c. (resp. l.s.c.) if for every V open in Y we have that $\{x \in X: F(x) \subseteq V\}$ (resp. $\{x \in X: F(x) \cap V \neq \emptyset\}$) is open in X. In order to define two more continuity notions for multifunctions, we need to introduce a mode of set convergence. Assume that Z is a subset of a normed space Y and τ is either w = weak topology on Y or s = strong (norm) topology on Y. If $\{A_n\}_{n \geq 1}$ are nonempty subsets of Z, we define

$$\tau - \overline{\lim_{n \to \infty}} A_n = \{z = \tau - \lim_{k \to \infty} z_{n_k}, z_{n_k} \in A_{n_k}, k \geq 1\}$$

and

$$\tau - \underline{\lim_{n \to \infty}} A_n = \{z = \tau - \lim_{n \to \infty} z_n, z_n \in A_n, n \geq 1\}.$$

When $\tau - \underline{\lim}_{n \to \infty} = \tau - \overline{\lim}_{n \to \infty} A_n = A$ we will say that A_n τ-converges to A in the Kuratowski sense and we write $A_n \to^{\tau K} A$ as $n \to \infty$. When $\mathrm{s} - \underline{\lim}_{n \to \infty} A_n = \mathrm{w} - \overline{\lim}_{n \to \infty} A_n = A$ we say that A_n converges to A in Kuratowski–Mosco sense and we write $A_n \to^{\mathrm{K\text{-}M}} A$ as $n \to \infty$. When (Z, τ) is metrizable then the τK-convergence is the classical Kuratowski convergence of sets (see Kuratowski [20]), while for the K–M convergence we refer to Mosco [24]. Let V be a locally convex space $F: V \to 2^V \backslash \{\emptyset\}$. We will say that $F(\cdot)$ is $\tau K -$ u.s.c. (resp. τK − l.s.c.) if for all $v_n \to v$ we have that $\tau - \overline{\lim}_{n \to \infty} F(v_n) \subseteq F(v)$ (resp. $F(v) \subseteq \tau - \underline{\lim}_{n \to \infty} F(v))$. Note that upper semicontinuity by inclusion (see Delahaye and Denel [9]) implies τK-upper semicontinuity and they are equivalent for $\tau = \mathrm{s}$. Also if $F(\cdot)$ is closed valued and u.s.c., then it is u.s.c. by inclusion hence τK-upper semicontinuous. The converse is false in general. On the other hand, if V is metrizable and $\tau = \mathrm{s}$, lower semicontinuity and sK-lower semicontinuity are always equivalent. Finally, $F(\cdot)$ is said to be K–M continuous if $v_n \to v \Rightarrow F(v_n) \to^{\mathrm{K\text{-}M}} F(v)$, while $F(\cdot)$ is said to be $d -$ u.s.c. if it is u.s.c. and $v \to d(y)_{F(v)}$ is continuous for all $y \in Y$.

3. Some Auxiliary Results

In this section we present some auxiliary results that we will need in the proof of the main theorems. Moreover those results are also interesting in their own as general results about multifunctions. So we present them in the outmost generality in which we were able to prove them.

The first result was obtained by the author in [25]. Assume that (Ω, Σ, μ) is a complete σ-finite measure space and X a separable Banach space.

THEOREM 3.1 [25]. *If $F: \Omega \to P_{\mathrm{wkc}}(X)$ is integrably bounded, then S_F^1 is a nonempty, weakly compact, and convex subset of $L_X^1(\Omega)$.*

Remarks. (1) If in addition (Ω, Σ, μ) is nonatomic and X^* separable, then we can have a converse of this result. Namely, if S_F^1 is w-compact, convex in $L_X^1(\Omega)$, then for all $\omega \in \Omega$, $F(\omega) \in P_{\mathrm{wkc}}(X)$ (for details see [26]).

(2) Our result generalizes in the context of separable Banach spaces Theorem 2 of Diestel [11].

(3) A useful consequence of that result is that for $F(\cdot)$ as in the theorem, $\int_\Omega F(\omega)\, d\mu(\omega) \in P_{\mathrm{wkc}}(X)$.

It is well known that if (Ω, Σ) is a measurable space, X a separable metric space, and $f: \Omega \times X \to \mathbb{R}$ a Carathéodory function (namely, for all $x \in X$ $f(\cdot, x)$ is measurable and for all $\omega \in \Omega$ $f(\omega, \cdot)$ is continuous) then $f(\cdot, \cdot)$ is jointly measurable. In the presence of separability of the process $f(\cdot, \cdot)$ and of a complete σ-finite measure space (Ω, Σ, μ) we can extend this result to the case where the paths are l.s.c. (or u.s.c.). The proof of this result can be found in [26].

LEMMA 3.1 [26]. *If $f: \Omega \times X \to \mathbb{R}$ is a function s.t.*

(1) *for all $x \in X$, $f(\cdot, x)$ is measurable,*

(2) *for all $\omega \in \Omega$, $f(\omega, \cdot)$ is l.s.c. and $f(\cdot, \cdot)$ is separable*

then $f(\cdot, \cdot)$ is jointly measurable.

With this lemma we can have a result on the superpositional measurability of a class of multifunctions. By X_{w} (resp. X_{s}) we denote the Banach space X with the weak (resp. strong) topology.

LEMMA 3.2. *If $F: \Omega \times X \to P_{\mathrm{fc}}(X)$ is a multifunction with bounded values s.t.*

(1) *for all $x \in X$, $F(\cdot, x)$ is measurable,*

(2) *for all $\omega \in \Omega$, $F(\omega, \cdot)$ is sw $-$ u.s.c. (i.e., u.s.c. from X_{s} into X_{w}) and is separable and if $x: \Omega \to X$ is measurable then $\omega \to F(\omega, x(\omega))$ is measurable. Furthermore if $F(\cdot, \cdot)$ has values in $P_{\mathrm{wck}}(X)$ and instead of* (2) *we have*

(2′) *for all* $\omega \in \Omega$, $F(\omega, \cdot)$ *is* sw − *l.s.c. from* X_s *into* X_w *then* $\omega \to F(\omega, x(\omega))$ *is measurable.*

Remarks. Note that any u.s.c. (resp. l.s.c.) multifunction is automatically sw − u.s.c. (resp. sw − l.s.c.).

For the next lemma assume that X is any metric space.

LEMMA 3.3. *If* $F\colon X \to P_f(X)$ *is u.s.c. then* $x \to d(x)_{F(x)}$ *is l.s.c.*

Proof. Let $\varepsilon > 0$. There exists $\delta > 0$ s.t. for all $y \in B_\delta(x)$, $F(y) \subseteq F(x) + B_{\varepsilon/4}(0)$. Assume $\delta \leqslant \varepsilon/4$. Take $z \in F(y)$ s.t. $\| y - z \| \leqslant d(y)_{F(y)} + \varepsilon/2$ and $v \in F(x)$ s.t. $\| z - v \| \leqslant \varepsilon/4$. Then

$$
\begin{aligned}
d(x)_{F(x)} - d(y)_{F(y)} - \frac{\varepsilon}{2} &\leqslant d(x)_{F(x)} - \| y - z \| \\
&\leqslant \| x - v \| - \| y - z \| \\
&\leqslant \| x - y \| + \| y - z \| + \| z - v \| - \| y - z \| \\
&= \| x - y \| + \| z - v \| \\
&\leqslant \delta + \frac{\varepsilon}{4} < \frac{\varepsilon}{2} \\
&\Rightarrow d(x)_{F(x)} \leqslant d(y)_{F(y)} + \varepsilon.
\end{aligned}
$$

Hence $x \to d(x)_{F(x)}$ is l.s.c. as claimed. Q.E.D.

Remark. If X is a reflexive Banach space and $F(\cdot)$ is wK-u.s.c. then the same result holds. The proof of that uses Theorem 2.2 of Tsukada [29]. Recall that every w-closed valued sw-upper semicontinuous multifunction is wK-upper semicontinuous. Hence in particular every u.s.c. multifunction with w-closed values is wK-u.s.c.

We will use the above lemma to obtain an interesting fixed point theorem for multifunctions. So assume that (Ω, Σ, μ) is a complete σ-finite measure space and X a separable Banach space, with a separable dual.

LEMMA 3.4. *If* $F\colon \Omega \times X \to P_{fc}(X)$ *is a multifunction s.t.*

(1) *for all* $x \in X$, $F(\cdot, x)$ *is measurable,*

(2) *for all* $\omega \in \Omega$, $F(\omega, \cdot)$ is u.s.c.,

(3) $F(\cdot, \cdot)$ has a wide sense fixed point, i.e., for all $\omega \in \Omega$ there exists $x(\omega) \in X$ *s.t.* $x(\omega) \in F(\omega, x(\omega))$ *and is separable then* $F(\cdot, \cdot)$ *admits a random fixed point, i.e., there exists* $x\colon \Omega \to X$ *measurable s.t. for all* $\omega \in \Omega$, $x(\omega) \in F(\omega, x(\omega))$.

Proof. Consider the multifunction $L: \Omega \to 2^X$ defined by $L(\omega) = \{x \in X: x \in F(\omega, x)\}$. From hypothesis (3) we know that for all $\omega \in \Omega$, $L(\omega) \neq \emptyset$. Rewrite $L(\cdot)$ as follows: $L(\omega) = \bigcap_{n \geqslant 1} \{x \in X: (x_n^*, x) \leqslant \sigma(x_n^*)_{F(\omega,x)}\}$ where $\{x_n^*\}_{n \geqslant 1}$ is a dense set in X^*. Because of the measurability of $F(\cdot, x)$, we know that $\omega \to \sigma(x_n^*)_{F(\omega,x)}$ is measurable. Also $x \to \sigma(x_n^*)_{F(\omega,x)}$ is u.s.c. since $F(\omega, \cdot)$ is and note that $\sigma(x_n^*)_{F(\cdot,\cdot)} - (x_n^*, \cdot)$ is separable since $F(\cdot, \cdot)$ is. So Lemma 3.1 tells us that $(\omega, x) \to \sigma(x_n^*)_{F(\omega,x)}$ is measurable for all $n \geqslant 1$. Hence $\operatorname{Gr} L = \bigcap_{n \geqslant 1} \{(\omega, x) \in \Omega \times X: \sigma(x_n^*)_{F(\omega,x)} - (x_n^*, x) \geqslant 0\} \in \Sigma \times B(X)$. Apply Aumann's selection theorem to find $x: \Omega \to X$ measurable s.t. for all $\omega \in \Omega$, $x(\omega) \in L(\omega)$. Clearly $x(\cdot)$ is the desired random fixed point for $F(\cdot, \cdot)$. Q.E.D.

For completeness, we present an alternative version of this result. Assume that (Ω, Σ, μ) is a complete σ-finite measure space and X a separable reflexive Banach space.

LEMMA 3.5. *If $F: \Omega \times X \to P_{fc}(X)$ is a multifunction s.t.*

(1) *for all $x \in X$, $F(\cdot, x)$ is measurable,*

(2) *for all $\omega \in \Omega$, $F(\omega, \cdot)$ is K–M continuous,*

(3) *$F(\cdot, \cdot)$ admits a wide sense fixed point*

then $F(\cdot, \cdot)$ has a random fixed point.

Proof. First note that because of hypothesis (1), for all $x \in X$, $\omega \to d(x)_{F(\omega,x)}$ is measurable. Next fix $\omega \in \Omega$ and consider the map $x \to d(x)_{F(\omega,x)}$. We claim that this map is continuous. To see that let $x_n \to^s x$. We have

$$\begin{aligned} &| d(x_n)_{F(\omega,x_n)} - d(x)_{F(\omega,x)} | \\ &\qquad \leqslant | d(x_n)_{F(\omega,x_n)} - d(x)_{F(\omega,x_n)} | + | d(x)_{F(\omega,x_n)} - d(x)_{F(\omega,x)} | \\ &\qquad \leqslant \| x_n - x \| + | d(x)_{F(\omega,x_n)} - d(x)_{F(\omega,x)} |. \end{aligned}$$

Since $x_n \to^s x \Rightarrow F(\omega, x_n) \to^{K\ M} F(\omega, x)$, Theorem 2.5 of Tsukada [29] tells us that $d(x)_{F(\omega,x_n)} \to d(x)_{F(\omega,x)}$. Thus $\lim_{n \to \infty} | d(x_n)_{F(\omega,x_n)} - d(x)_{F(\omega,x)} | = 0$, which shows that $(\omega, x) \to d(x)_{F(\omega,x)}$ is Carathéodory, hence jointly measurable. Then $L(\omega) = \{x \in X: d(x)_{F(\omega,x)} = 0\}$ is graph measurable and so Aumann's theorem produces the desired random fixed point. Q.E.D.

4. DETERMINISTIC RESULTS

In this section we prove two new existence results for deterministic differential inclusions in Banach spaces. Those two results together with the material of Section 3 will be used in the proof of the theorems in Section 5. Moreover they are interesting in their own, because they are new existence results for differential inclusions in infinite-dimensional Banach spaces, where very little work was done until now. Most of the work existing in the literature is in $\mathbb{R}^n$ (see Aubin and Cellina [1] and references therein).

We are going to examine the Cauchy problem

$$\left\{\begin{array}{l}\dot{x}(t) \in F(t, x(t)) \\ x(0)=x_0\end{array}\right\} \tag{D}$$

where $t \in [0, T]$ $(T<\infty)$ and $x_0 \in X =$ separable Banach space. On $[0, T]$ we consider the Lebesgue measure $\lambda(\cdot)$ and the σ-field Σ_T of Lebesgue measurable sets. By a solution of (D) we understand an absolutely continuous function x: $[0, T] \to X$ s.t. $\dot{x}(t) \in F(t, x(t))$ a.e. and $x(0)=x_0$. From now on for economy in the notation, when no confusion is possible, we will write T to denote $[0, T]$.

Our first existence result involves u.s.c. orientor fields and generalizes Theorem 4.2 of Davy [7] and Theorem VI-8 of Castaing and Valadier [4]. Assume that T and X are as above.

THEOREM 4.1. *If F: $T \times X \to P_{fc}(X)$ is a multifunction s.t.*

(1) *for all $x \in X$, $F(\cdot, x)$ is measurable and $F(t, x) \subseteq G(t)$ a.e. where G: $T \to P_{wkc}(X)$ is integrably bounded,*

(2) *for all $t \in T$, $F(t, \cdot)$ is u.s.c. from X_w into X_w then (D) admits a solution.*

Proof. Let $W = \{x(\cdot) \in C_X(T)$: $x(t) = x_0 + \int_0^t f(s)\, ds$, $t \in T$ and $f(\cdot) \in S_G^1\}$. First we will show that W is a compact subset of $C_{X_w}(T)$. Note that for all $t \in T$ we have $x(t) \in x_0 + \int_0^t G(s)\, ds$. From Theorem 3.1 we know that $x_0 + \int_0^t G(s)\, ds \in P_{wkc}(X)$ for all $t \in T$. Hence we have that for all $t \in T \{x(t)$: $x(\cdot) \in W\}$ is relatively w-compact in X. Next from the absolute continuity of the Lebesgue integral we have that given $\varepsilon > 0$ there exists $\delta > 0$ s.t. if $t_1, t_2 \in T$, $t_1 \leqslant t_2$, and $t_2 - t_1 < \delta$ then $\int_{t_1}^{t_2} |G(s)|\, ds < \varepsilon$. So for $x(\cdot) \in W$ we have

$$\begin{aligned} \|x(t_2) - x(t_1)\| &= \left\| x_0 + \int_0^{t_2} f(s)\, ds - x_0 - \int_0^{t_1} f(s)\, ds \right\| \\ &\leqslant \int_{t_1}^{t_2} \|f(s)\|\, ds < \varepsilon. \end{aligned}$$

This shows that W is an equicontinuous set and so a fortiori weakly equicontinuous. Now if we show that W is closed in $C_{X_w}(T)$, then the

Arzelà–Ascoli theorem will tell us that this set is compact in $C_{X_w}(T)$. So let $\{x_a(\cdot)\} \subseteq W$ be a net s.t. $x_a(\cdot) \to^{C_{X_w}} x(\cdot)$. Then for every a we have $x_a(t) = x_0 + \int_0^t f_a(s)\,ds$ with $f_a(\cdot) \in S_G^1$. Once again we use Theorem 3.1 to get a subnet $\{f_b(\cdot)\}$ s.t. $f_b(\cdot) \to^{w-L_X^1} f(\cdot) \in S_G^1$. Hence for all $t \in T$ we have

$$x_b(t) = x_0 + \int_0^t f_b(s)\,ds \xrightarrow{\ w\ } x_0 + \int_0^t f(s)\,ds.$$

But we already know that $x_b(\cdot) \to^{C_{X_w}} x(\cdot)$, Thus for all $t \in T$ we have $x(t) = x_0 + \int_0^t f(s)\,ds$ with $f(\cdot) \in S_G^1$, i.e., $x(\cdot) \in W$. Therefore W is closed in $C_{X_w}(T)$ and the Arzelà–Ascoli theorem compact in $C_{X_w}(T)$.

Next consider the multifunction L: $W \to 2^{C_X(T)}$ defined by $L(x) = \{y(\cdot) \in C_X(T)$: $y(t) = x_0 + \int_0^t f(s)\,ds,\ t \in T,\ f(\cdot) \in S^1_{F(\cdot, x(\cdot))}\}$. Approximating $x(\cdot)$ by simple functions, we can easily see that $S^1_{F(\cdot, x(\cdot))} \neq \varnothing$. So for all $x(\cdot) \in W$, $L(x) \neq \varnothing$. Also note that $L(\cdot)$ is closed and convex valued. Since X is separable, $L_X^1(T)$ is separable and so the w-compact set S_G^1 with the weak topology is metrizable (see Dunford and Schwartz [12, Theorem 3, p. 434]). Hence $\{x_0\} \times (S_G^1, w)$ is metrizable. But W is isomorphic to $\{x_0\} \times (S_G^1, w)$. Therefore W is a metrizable subset of $C_{X_w}(T)$. Next let $(x_n, y_n) \to^{W \times W} (x, y)$ with $y_n \in L(x_n)$, $n \geqslant 1$. Then we have $y_n(t) = x_0 + \int_0^t f_n(s)\,ds$, $t \in T$, $f_n(\cdot) \in S^1_{F(\cdot, x_n(\cdot))}$. Theorem 3.1 tells us that by passing to a subsequence, if necessary, we may assume that $f_n(\cdot) \to^{w-L_X^1} f(\cdot) \in S_G^1$ as $n \to \infty$. Invoking Mazur's theorem we can find $z_n(\cdot) \in \operatorname{conv} \bigcup_{k \geqslant n} f_k(\cdot)$ s.t. $z_n(\cdot) \to^{s-L_X^1} f(\cdot)$ as $n \to \infty$ and once again by passing to a subsequence, if necessary, we may assume that $z_n(t) \to^{s} f(t)$ as $n \to \infty$ for $t \in T \backslash N$, $\lambda(N) = 0$. Fix $t \in T \backslash N$. Because by hypothesis $F(t, \cdot)$ is ww – u.s.c. we know that given $U =$ convex weak neighborhood of the origin in X, there exists $n \geqslant 1$ s.t. for $k \geqslant n$ we have $F(t, x_k(t)) \subseteq F(t, x(t)) + U \Rightarrow \overline{\operatorname{conv}} \bigcup_{k \geqslant n} F(t, x_k(t)) \subseteq F(t, x(t)) + U \Rightarrow f(t) \in F(t, x(t)) + U$. But U was arbitrary. So $f(t) \in F(t, x(t))$ for all $t \in T \backslash N \Rightarrow f(\cdot) \in S^1_{F(\cdot, x(\cdot))}$. Thus $L(\cdot)$ has closed graph. This then, by Delahaye and Denel [9], means that $L(\cdot)$ is u.s.c. Applying Kakutani's fixed point theorem we deduce that there exists $\hat{x}(\cdot) \in W$ s.t. $\hat{x}(\cdot) \in L(\hat{x}(\cdot))$. Clearly $\hat{x}(\cdot)$ solves (D). Q.E.D.

We also have an existence result for sK-lower semicontinuous, nonconvex orientor fields. Our result generalizes Theorem 2 of Bressan [3] and Theorem 1 of Lojasiewicz [22].

THEOREM 4.2. *If* F: $T \times X \to P_{\mathrm{wk}}(X)$ *is a multifunction s.t.*

(1) $(t, x) \to F(t, x)$ *is measurable and* $F(t, x) \subseteq G(t)$ *a.e. where* G: $T \to P_{\mathrm{wkc}}(X)$ *is integrable bounded,*

(2) *for all* $t \in T$, $F(t, \cdot)$ *is* $sK-l.s.c.$ *from* X_w *into* X *and* $S^1_{F(\cdot, x(\cdot))} \neq \varnothing$ *for all* $x(\cdot) \in C_X(T)$ *then* (D) *admits a solution.*

Proof. Again consider the set $W = \{x(\cdot) \in C_X(T): x(t) = x_0 + \int_0^t f(s)\, ds, t \in T, f(\cdot) \in S_G^1\}$. We have already seen in the proof of Theorem 4.1 that W is a compact metrizable subset of $C_{X_w}(T)$. Next consider the multifunction $R: W \to P_f(T)$ defined by $R(x) = S^1_{F(\cdot,x(\cdot))}$. Let $A \in \Sigma$ and $y_1(\cdot), y_2(\cdot) \in R(x)$. Set $y(\cdot) = \chi_A y_1(\cdot) + \chi_{A^c} y_2(\cdot)$. It is easy to see that $y(\cdot) \in S^1_{F(\cdot,x(\cdot))}$. So $R(\cdot)$ has decomposable values. We will show that $R(\cdot)$ is l.s.c. from $W \subseteq C_{X_w}(T)$ into $L^1_X(T)$. So let $\{x_n(\cdot)\}_{n \geqslant 1} \subseteq W$ s.t. $x_n(\cdot) \to^{C_{X_w}} x(\cdot)$ as $n \to \infty$. Let $y(\cdot) \in R(x)$ and $A_n(t) = \{z \in F(t, x_n(t)): \|z - y(t)\| = d(y(t))_{F(t,x_n(t))}\} n \geqslant 1$. As before we can show that $\operatorname{Gr} A_n \in \Sigma \times B(X)$ and so we can find $y_n: T \to X$ measurable s.t. for all $t \in T$, $y_n(t) \in A_n(t)$, $n \geqslant 1$. Hence for all $t \in T$, $n \geqslant 1$ $d(y(t))_{F(t,x_n(t))} = \| y_n(t) - y(t) \|$. Also from Theorem 2.2 of [29] we can easily see that $x \to d(y)_{F(t,x)}$ is w-sequentially u.s.c. We will show that $\| y_n(t) - y(t) \| \to 0$ a.e. as $n \to \infty$. To see that note

$$0 = d(y(\cdot))_{S^1_{F(\cdot,x(\cdot))}} = \inf_{g(\cdot) \in S^1_{F(\cdot,x(\cdot))}} \int_T \| y(t) - g(t) \|\, dt$$

$$= \int_T \inf_{z \in F(t,x(t))} \| y(t) - z \|\, dt = \int_T d(y(t))_{F(t,x(t))}\, dt$$

$$\Rightarrow d(y(t))_{F(t,\,x(t))} = 0 \qquad \text{a.e.} \Rightarrow \| y(t) - y_n(t) \| \to 0 \text{ a.e. as } n \to \infty.$$

Then $y_n(\cdot) \to^{s - L^1_X} y(\cdot)$ and so $y(\cdot) \in s - \underline{\lim}_{n \to \infty} R(x_n)$. Thus we have proved that $R(x) \subseteq s - \underline{\lim}_{n \to \infty} R(x_n)$. Because W is metrizable from Delahaye and Denel [9] we conclude that $R(\cdot)$ is l.s.c. Now apply Theorem 3.1 of Fryszkowski [15] to get $r: W \to L^1_X(T)$ continuous s.t. $r(x) \in R(x)$ for all $x \in W$. Set $\phi(x)(t) = x_0 + \int_0^t r(x)(s)\, ds$, $t \in T$. Note that $\phi(x)(\cdot)$ is absolutely continuous. Also $x \to \phi(x)(\cdot)$ is continuous from W into W. Tichonoff's fixed point theorem guarantees the existence of $\hat{x}(\cdot) \in W$ s.t. $\hat{x} = \phi(\hat{x})$. It is easy to see that $\hat{x}(\cdot)$ solves (D). Q.E.D.

Remarks. (1) If $F(t, \cdot)$ is ws $-$ l.s.c. it is automatically sK $-$ l.s.c. from X_w into X (see [9]).

(2) Lower semicontinuous, nonconvex orientor fields appear often in control theory. Namely, suppose $\dot{x}(t) = f(t, x(t), u)$ describes the dynamics of the controlled system, with $u \in U$ being the control variable. As we said in the introduction, if we set $F(t, x) = \bigcup_{u \in U} f(t, x, u)$, then the system is described by the differential inclusion $\dot{x}(t) \in F(t, x(t))$. We may ask whether we can find solutions of that inclusion which move only through the extreme points of $F(\cdot, \cdot)$, namely, we are asking the question whether $\dot{x}(t) \in \operatorname{ext} F(t, x(t))$ has a solution. In the linear case research in that direction led to the celebrated bang-bang principle (see Cesari [5] and Hermes and LaSalle [16]). If on X we consider the weak topology, U is a compact topological space, and $f(\cdot, x, u)$is measurable while $f(t, \cdot, \cdot)$ is continuous,

then we can show that $F(\cdot, x)$ is measurable, $F(t, \cdot)$ is Hausdorff continuous from X_w into $P_k(X)$ and so for all $x \in X$, $\text{ext}\, F(t, x) \neq \emptyset$ and $x \to \text{ext}\, F(t, x)$ is ws $-$ l.s.c. Furthermore if we assume that $f(t, x, U)$ is convex, Theorem 1 of Himmelberg and Van Vleck [18] tells us that $t \to \text{ext}\, F(t, x)$ is measurable. Finally, if for all $x \in X$ and all $u \in U$, $f(t, x, u) \in G(t)$ where G: $T \to P_{wkc}(X)$ is integrably bounded, then we can apply Theorem 4.2 and conclude the existence of a solution for $x(t) \in \overline{\text{ext}}^w F(t, x)$.

(3) If the domain of $F(\cdot, \cdot)$ is the set $T \times B_r(x_0)$, where $B_r(x_0) = \{x \in X: \|x - x_0\| \leq r\}$, then local versions of both theorems are valid.

Additional results about (D) can be found in [27].

5. Stochastic Results

In this section we examine random differential inclusions. So the Cauchy problem that we are going to study has the form

$$\begin{cases} \dot{x}(\omega, t) \in F(\omega, t, x(\omega, t)) \\ x(\omega, 0) = x_0(\omega) \end{cases} \qquad (SP)$$

where ω belongs to an underlying complete probability space (Ω, Σ, μ) and x_0: $\Omega \to X$ is measurable. As before $[0, T]$ $(T < \infty)$ and X is a separable Banach space. By a solution of (SP) we understand a stochastic process x: $\Omega \times T \to X$, which has continuous sample paths and satisfies (SP) with probability one for almost all $t \in T$. Recall that $x(\cdot, \cdot)$ being a stochastic process is by definition measurable in ω for all $t \in T$.

We start with a result for u.s.c. random orientor fields. Assume that X is a finite-dimensional Banach space.

THEOREM 5.1. *If F: $\Omega \times T \times X \to P_{fc}(X)$ is a multifunction s.t.*

(1) *$F(\cdot, \cdot, \cdot)$ is jointly measurable and for all $x \in X$, $F(\omega, t, x) \subseteq G(\omega, t)$ a.e. for all $\omega \in \Omega$, where G: $\Omega \times T \to P_{kc}(X)$ is jointly measurable and integrably bounded in t,*

(2) *for all $(\omega, t) \in \Omega \times T$, $F(\omega, t, \cdot)$ is $d-$u.s.c. then (SP) admits a solution.*

Proof. Consider the multifunction R: $\Omega \times C_X(T) \to 2^{C_X(T)}$ defined by $R(\omega, x) = \{y(\cdot) \in C_X(T)$: $y(t) = x_0(\omega) + \int_0^t f(s)\, ds$, $t \in T$, $f(\cdot) \in S^1_{F(\omega, \cdot, x(\cdot))}\}$. Because of hypothesis (2) we can see that $S^1_{F(\omega, \cdot, x(\cdot))} \neq \emptyset$ and so $R(\omega, x) \neq \emptyset$. Also it is easy to see using the w-compactness of $S^1_{F(\omega, \cdot, x(\cdot))}$ that $R(\omega, x)$ is closed and convex.

Now fix $x(\cdot)\in C_X(T)$ and consider $\omega\to R(\omega, x)$. We claim that it is measurable. Let $\Phi(\omega, t)=x_0(\omega)+\int_0^t F(\omega, s, x(s))\, ds$. For every $x^*\in X^*$ we have

$$\sigma(x^*)_{\Phi(\omega,t)}=(x^*, x_0(\omega))+\int_0^t \sigma(x^*)_{F(\omega,s,x(s))}\, ds.$$

But $(\omega, s)\to\sigma(x^*)_{F(\omega,s,x(s))}$ is measurable and integrable in s. So $\omega\to\int_0^t \sigma(x^*)_{F(\omega,s,x(s))}\, ds$ is a random variable and thus $\omega\to\sigma(x^*)_{\Phi(\omega,t)}$ is measurable. Also $t\to\sigma(x^*)_{\Phi(\omega,t)}$ is absolutely continuous. Therefore $(\omega, t)\to\sigma(x^*)_{\Phi(\omega,t)}$ is a Carathéodory function and so it is $\Sigma\times B(T)$-measurable. Since $\Phi(\cdot,\cdot)$ has values in $P_{kc}(X)$, Theorem III-37 of Castaing and Valadier [4] tells us that $(\omega, t)\to\Phi(\omega, t)$ is $\Sigma\times\Sigma_T$-measurable. Now rewrite $R(\omega, x)$ as $R(\omega, x)=R_1(\omega)\cap R_2(\omega)$ where $R_1(\omega)=\{y(\cdot)\in C_X(T)\colon\ d(y(\cdot))_{S^1_{\Phi(\omega,\cdot)}}=0\}$, and $R_2(\omega)=\{y(\cdot)\in C_X(T)\colon y(0)=x_0(\omega)\}$. Clearly $S^1_{\Phi(\omega,\cdot)}\neq\varnothing$. Note that

$$\begin{aligned} d(y(\cdot))_{S^1_{\Phi(\omega,\cdot)}} &=\inf_{\phi(\cdot)\in S^1_{\Phi(\omega,\cdot)}}\int_T \| y(t)-\phi(t)\|\, dt \\ &=\int_T \inf_{z\in\Phi(\omega,t)}\| y(t)-z\|\, dt \\ &=\int_T d(y(t))_{\Phi(\omega,t)}\, dt. \end{aligned}$$

But $(\omega, t)\to d(z)_{\Phi(\omega,t)}$ is $\Sigma\times\Sigma_T$-measurable and $z\to d(z)_{\Phi(\omega,t)}$ is continuous. Hence $(\omega, t, z)\to d(z)_{\Phi(\omega,t)}$ is a Carathéodory function and so is superpositionally measurable $\Rightarrow (\omega, t)\to d(y(t))_{\Phi(\omega,t)}$ is $\Sigma\times\Sigma_T$-measurable. Then as before we get that $\omega\to d(y(\cdot))_{S^1_{\Phi(\omega,\cdot)}}$ is measurable. So we have that $(\omega, y(\cdot))\to d(y(\cdot))_{S^1_{\Phi(\omega,\cdot)}}$ is a Carathéodory function and so $\Sigma\times B(C_X(T))$-measurable. Therefore $\mathrm{GR}\, R_1=\{(\omega, y(\cdot))\in\Omega\times C_X(T)\colon d(y(\cdot))_{S^1_{\Phi(\omega,\cdot)}}=0\}\in\Sigma\times B(C_X(T))$.

For $R_2(\cdot)$ we have $R_2(\omega)=\{y(\cdot)\in C_X(T)\colon\ y(0)=x_0(\omega)\}=\{y(\cdot)\in C_X(T)\colon e_0(y(\cdot))=x_0(\omega)\}$ where $e_0(\cdot)$ is the evaluation map at 0. We know that this is continuous. So $(\omega, y(\cdot))\to e_0(y(\cdot))-x_0(\omega)$ is Carathéodory, hence jointly measurable. Thus $\mathrm{Gr}\, R_2\in\Sigma\times B(C_X(T))$. Therefore we get that $\mathrm{Gr}\, R(\cdot, x)=\mathrm{Gr}\, R_1\cap\mathrm{Gr}\, R_2\in\Sigma\times B(C_X(T))$, which together with the fact that $R(\cdot, x)$ is closed valued implies measurability.

Next we will show that for all $\omega\in\Omega$, $R(\omega,\cdot)$ is u.s.c. Note that for all $\omega\in\Omega$, $R(\omega,\cdot)$ is a subset of the compact set $W(\omega)=\{(y(\cdot)\in C_X(T)\colon y(t)=x_0(\omega)+\int_0^t f(s)\, ds,\quad t\in T,\quad f(\cdot)\in S^1_{G(\omega,\cdot)}\}$. Let $x_n(\cdot)\to^{C_X} x(\cdot)$ and $y_n(\cdot)\to^{C_X} y(\cdot)$ as $n\to\infty$ with $y_n(\cdot)\in R(\omega, x_n)$. We need to show that

$y(\cdot) \in R(\omega, x)$. From the definition of $R(\cdot,\cdot)$ for every $n \geq 1$ and for every $t \in T$, we have $y_n(t) = x_0(\omega) + \int_0^t f_n(s)\, ds$ with $f_n(\cdot) \in S^1_{F(\omega,\cdot,x_n(\cdot))}$. But $S^1_{F(\omega,\cdot,x_n(\cdot))} \subseteq S^1_{G(\omega,\cdot)}$ for all $n \geq 1$ and all $\omega \in \Omega$ and the latter set is w-compact in $L^1_X(T)$. So we may assume that $f_n(\cdot) \to^{w-L^1_X} f(\cdot)$ as $n \to \infty$. Mazur's lemma provides us functions $z_n(\cdot) \in \operatorname{conv} \bigcup_{k \geq n} f_k(\cdot)$ s.t. $z_n(t) \to^s f(t)$ for all $t \in T \backslash N$, $\lambda(N) = 0$. Let $t \in T \backslash N$. Then given $\varepsilon > 0$ we can find $n_0 \geq 1$ s.t. for $n \geq n_0$ we have $F(\omega, t, x_n(t)) \subseteq F(\omega, t, x(t)) + \varepsilon B_1$ (B_1 being the unit ball in X) $\Rightarrow \operatorname{conv} \bigcup_{n \geq n_0} F(\omega, t, x_n(t)) \subseteq F(\omega, t, x(t)) + \varepsilon B_1 \Rightarrow f(t) \in F(\omega, t, x(t)) + \varepsilon B_1$. Let $\varepsilon \downarrow 0$ and we finally have that $f(\cdot) \in S^1_{F(\omega,\cdot,x(\cdot))}$. Hence $y(\cdot) \in R(\omega, x)$ and so $x \to R(\omega, x)$ is u.s.c.

Now we will show that $R(\cdot,\cdot)$ is separable. Consider $\{x_n(\cdot)\}_{n \geq 1}$ a dense subset of $C_X(T)$, let $r \in Q$, and let $\hat{U}$ be the countable family of the sets $B_r(x_n) = \{z(\cdot) \in C_X(T) \colon \|z(\cdot) - x_n(\cdot)\| \leq r\}$, $[\overline{B_r(x_n)}]^c$ and their finite intersections. For any closed $K \in \hat{U}$, let D_K be a countable dense set in K and let $D = \bigcup_{K \in \hat{U}} D_K$. We need to show that $\overline{R(\omega, K \cap D)} = \overline{R(\omega, K)}$. Let $y(\cdot) \in R(\omega, K)$. Then $y(t) = x_0(\omega) + \int_0^t f(s)\, ds$, $f(\cdot) \in S^1_{F(\cdot,x(\cdot))}$, $t \in T$. Let $x_n(\cdot) \in K \cap D$ s.t. $x_n(\cdot) \to^{C_X} x(\cdot)$. Using Aumann's theorem we can find $f_n(\cdot) \in S^1_{F(\omega,\cdot,x_n(\cdot))}$ s.t. $\|f_n(t) - f(t)\| = d(f(t))_{F(\omega,t,x_n(t))} \to 0$ as $n \to \infty$. Let $y_n(t) = x_0(\omega) + \int_0^t f_n(s)\, ds$. Then $\|y_n(t) - y(t)\| \leq \int_T \|f_n(s) - f(s)\|\, ds \to 0 \Rightarrow \|y_n(\cdot) - y(\cdot)\|_\infty \to 0 \Rightarrow R(\cdot,\cdot)$ is separable. From Theorem 4.1 we know that for every $\omega \in \Omega$, $(SP)_\omega$ has a solution and this is a wide sense fixed point for $R(\cdot,\cdot)$. Now apply Lemma 3.4 to get that there exists ϕ: $\Omega \to C_X(T)$ measurable s.t. for all $\omega \in \Omega$ and we have $\phi(\omega) \in R(\omega, \phi(\omega))$. Set $\phi(\omega)(\cdot) = x(\omega, \cdot)$. Then Proposition 4.2 of Itoh [19] tells us that $x(\cdot,\cdot)$ is Carathéodory and clearly solves (SP). Q.E.D.

Remark. An alternative proof can be based on Theorem 16 of Engl [13]. In this case we consider $R(\cdot,\cdot)$ defined on Gr W. The only thing that we need to check to see that this approach works is to show that $W(\cdot)$ is separably determined. It suffices to show that $\operatorname{int} W(\omega) \neq \emptyset$ for all $\omega \in \Omega$. By considering, if necessary, $G(\omega, t) + B_1$ we may assume that $\operatorname{int} G(\omega, t) \neq \emptyset$. Let $g(\cdot) \in S^1_{\operatorname{int} G(\omega,\cdot)}$. Then we claim that $y_g(\cdot) = x_0(\omega) + \int_0^{\cdot} g(s)\, ds \in \operatorname{int} W(\omega)$. If not we can find $y_n(\cdot) \in C_X(T)$ s.t. $y_n(\cdot) \to^{C_X} y(\cdot)$ and $y_n(t_n) - x_0(\omega) \notin \int_0^{t_n} G(\omega, s)\, ds$ for some $t_n \in T$. By passing to a subsequence, if necessary, we may assume that $t_n \to t_0$. Then $y_n(t_n) - x_0(\omega) \to y(t_0) - x_0(\omega)$ and $\int_0^{t_n} G(\omega, s)\, ds \to^h \int_0^{t_0} G(\omega, s)\, ds$. Because $y(t_0) - x_0(\omega) \in \int_0^{t_0} \operatorname{int} G(\omega, s)\, ds = \operatorname{int} \int_0^{t_0} G(\omega, s)\, ds$, we have that for $n \geq n_1$, $y_n(t_n) - x_0(\omega) \in \operatorname{int} \int_0^{t_0} G(\omega, s)\, ds$. But note that

$$\begin{aligned} d_{\operatorname{bd} \int_0^{t_n} G(\omega,s)}(y_n(t_n) - x_0(\omega)) &= d_{\int_0^{t_n} G(\omega,s)}(y_n(t_n) - x_0(\omega)) \\ &\leq h\left(\int_0^{t_n} G(\omega, s)\, ds, \int_0^{t_0} G(\omega, s)\, ds\right) \to 0 \end{aligned}$$

as $n \to \infty$. Thus using Proposition 2.1 of [8] we get that

$$d_{\mathrm{bd}\int_0^{t_0} G(\omega,s)}(y(t_0)-x_0(\omega))=0 \Rightarrow y(t_0)-x_0(\omega)\in \mathrm{bd}\int_0^{t_0} G(\omega, s)\, ds,$$

a contradiction.

We will use Theorem 5.1 to prove an existence theorem for nonclosed valued orientor fields. Again X is finite dimensional. We will need the following lemma that partially generalizes Proposition 4.2 of [19]. Note that by $C(X)$ we denote the space of continuous functions from X into itself endowed with the compact-open topology.

LEMMA 5.1. *Let $f\colon \Omega \times X \to X$. Then $f(\cdot,\cdot)$ is a Carathéodory function if and only if $r\colon \Omega \to C(X)$ defined by $r(\omega)(\cdot)=f(\omega,\cdot)$ is measurable.*

Proof. First assume that $f(\cdot,\cdot)$ is Carathéodory. Let B be a basis element for the c – o topology on $C(X)$. Then there exist $K \subseteq X$ compact and $V \subseteq X$ open s.t. $B=\{g(\cdot)\in C(X)\colon g(K)\subseteq V\}$. We need to show that $r^{-1}(B)\in \Sigma$. If $\{z_n\}_{n\geqslant 1}$ is dense in K, then $r^{-1}(B)=\{\omega\in\Omega\colon r(\omega)(\cdot)\in B\}=\{\omega\in\Omega\colon r(\omega)(K)\subseteq V\}=\{\omega\in\Omega\colon f(\omega,K)\subseteq V\}=\bigcap_{n\geqslant 1}\{\omega\in\Omega\colon f(\omega,z_n)\in K\}\in\Sigma$. Now assume that $r(\cdot)$ is measurable. Let $(r, \mathrm{id})\colon \Omega\times X\to C(X)\times X$ be defined by $(r,\mathrm{id})(\omega,x)=(r(\omega)(\cdot),x)$. Clearly this is measurable. Let $e(\cdot,\cdot)$ be the evaluation map on $C(X)\times X$. This is continuous. Then $u(\cdot,\cdot)=e(r,\mathrm{id})(\cdot,\cdot)$ is Carathéodory. But $u(\cdot,\cdot)\equiv f(\cdot,\cdot)$. So $f(\cdot,\cdot)$ is Carathéodory. Q.E.D.

THEOREM 5.2. *If $F\colon \Omega\times T\times X\to 2^X\backslash\{\phi\}$ has solid, convex values and*

(1) for all $x\in X$, $F(\cdot,\cdot,x)$ is jointly measurable and for all $\omega\in\Omega$, $F(\omega,t,x)\subseteq G(\omega,t)$ a.e. where $G\colon \Omega\to P_{\mathrm{kc}}(X)$ is integrably bounded,

(2) for all $(\omega,t)\in\Omega\times T$, $F(\omega,t,\cdot)$ is continuous for the Hausdorff pseudometric

then (SP) has a solution.

Proof. Consider $R\colon \Omega\times T\to 2^{C(X)}$ defined by $R(\omega,t)=\{g(\cdot)\in C(X)\colon g(x)\in \mathrm{int}\, F(\omega,t,x)$ for all $x\in X\}$. Note that hypothesis (2) in particular implies that $F(\omega,t,\cdot)$ is l.s.c. Since $\overline{F(\omega,t,\cdot)}=\overline{\mathrm{int}\, F(\omega,t,\cdot)}$ we deduce using Proposition 2.3 of Michael [23] that $x\to \mathrm{int}\, F(,t,x)$ is l.s.c. So Theorem 3.1‴ of [23] tells us that for all (ω,t), $R(\omega,t)\neq\varnothing$. Let $C_{\overline{F(\omega,t,\cdot)}}$ be the continuous selectors of $\overline{F(t,\omega,\cdot)}$. Clearly this is a measurable multifunction of (ω,t). Note that $R(\omega,t)=\{g(\cdot)\in C(X)\colon d_{\mathrm{bd}\,F(\omega,t,x)}(g(x))>0$, $x\in X\}\cap C_{\overline{F(\omega,t,\cdot)}}$. From Theorem 4.6 of [17] we know that $(\omega,t)\to \mathrm{bd}\, F(\omega,t,x)$ is measurable. Also note that $\mathrm{bd}\, F(\omega,t,x)=\mathrm{bd}\,\overline{F(\omega,t,x)}$.

So $\overline{R(\omega, t)} = \{g(\cdot) \in C(X): \ d_{\text{bd}\,\overline{F(\omega,t,x)}}(g(x)) > 0, \ \ x \in X\} \cap C_{\overline{F(\omega,t,\cdot)}}$. But $x \to \overline{F(\omega, t, x)}$ is Hausdorff continuous. Then Proposition 2.1 of [8] tells us that $x \to \text{bd}\, F(\omega, t, x) = \text{bd}\, \overline{F(\omega, t, x)}$ is Hausdorff continuous.

Now we claim that for all $(\omega, t) \in \Omega \times T$ and for $g(\cdot) \in C(X)$, $x \to d_{\text{bd}\,(\omega,t,x)}(g(x))$ is continuous. Let $\varepsilon > 0$ be given and let $\delta(t, x) > 0$ be such that if $\|x - y\| < \delta$ then $h(\text{bd}\, F(\omega, t, x), \ \text{bd}\, F(\omega, t, y)) < \varepsilon/2$ and $\|g(x) - g(y)\| < \varepsilon/2$. Then using the triangle inequality for $h(\cdot,\cdot)$ and the Lipschitzness of the distance function, we get that when $\|x - y\| < \delta$

$$
\begin{aligned}
|d_{\text{bd}\,F(\omega,t,x)}&(g(x)) - d_{\text{bd}\,F(\omega,t,y)}(g(y))| \\
&\leqslant h(\text{bd}\, F(\omega, t, x), \text{bd}\, F(\omega, t, y)) + \|g(x) - g(y)\| < \varepsilon.
\end{aligned}
$$

So $x \to d_{\text{bd}\,F(\omega,t,x)}(g(x))$ is continuous. Now consider the map L: $\Omega \times T \times C(X) \to C_{\mathbb{R}}(X)$ defined by $(\omega, t, g(\cdot)) \to d_{\text{bd}\,F(\omega,t,\cdot)}(g(\cdot))$. Using Lemma 5.1 we have that for fixed $g(\cdot) \in C(X)$ the map $(\omega, t) \to d_{\text{bd}\,F(\omega,t,\cdot)}(g(\cdot))$ is measurable form $\Omega \times T$ into $C_{\mathbb{R}}(X)$. Also using once more the fact that the distance function is Lipschitz and because on $C(X)$ and $C_{\mathbb{R}}(X)$ the c – o topology and the topology of uniform convergence on compact sets coincide, we get that the map $g(\cdot) \to d_{\text{bd}\,F(\omega,t,\cdot)}(g(\cdot))$ is continuous from $C(X)$ into $C_{\mathbb{R}}(X)$. Thus $L(\cdot,\cdot,\cdot)$ is Carathéodory. From Theorems 1 and 3 of Kuratowski [21, p. 94] we know that $C(X)$ and $C_{\mathbb{R}}(X)$ are separable metric spaces. So $L(\cdot,\cdot,\cdot)$ is $\Sigma \times \Sigma \times B(C(X))$-measurable. Apply Aumann's selection theorem to find r: $\Omega \times T \to C(X)$ measurable s.t. $r(\omega, t)(\cdot) \in R(\omega, t)$. Then by Lemma 5.1 $r(\omega, t)(\cdot) = f(\omega, t, \cdot)$ is Carathéodory. Now consider $\dot{x}(t) = f(\omega, t, x(t))$, $x(\omega, 0) = x_0(\omega)$. By Theorem 5.1 this has a solution $x(\cdot,\cdot)$, which is also a solution of (SP). Q.E.D.

Finally, we have an infinite-dimensional version of Theorem 5.1. Here X is a separable, reflexive Banach space.

THEOREM 5.3. *If F: $\Omega \times T \times X \to P_{\text{wkc}}(X)$ is a multifunction s.t.*

(1) *$F(\cdot,\cdot,\cdot)$ is jointly measurable and for all $x \in X$, all $\omega \in \Omega$, $F(\omega, t, x) \subseteq G(\omega, t)$ a.e. where G: $\Omega \times T \to P_{\text{wkc}}(X)$ is integrably bounded,*

(2) *for all $(\omega, t) \in \Omega \times T$, $F(\omega, t, \cdot)$ is K–M-continuous from X_w into X*

then (SP) has a solution.

Proof. Consider the multifunction R: $\Omega \to 2^{C_X(T)}$ defined by $R(\omega) = \{x(\cdot) \in C_X(T): \ x(t) = x_0(\omega) + \int_0^t f(s)\, ds, \ \ t \in T, \ \ S^1_{F(\omega,\cdot,x(\cdot))}\}$. From Theorem 4.2 we know that for all $\omega \in \Omega$, $R(\omega) \neq \varnothing$. Let

$$
\phi(\omega, t, x(\cdot)) = d(x(t))_{x_0(\omega) + \int_0^t F(\omega,s,x(s))\, ds}.
$$

Then we have $R(\omega)=\{x(\cdot)\in C_X(T)\colon \phi(\omega,t,x(\cdot))=0\}$. Note that the multifunction $(\omega,t)\to x_0(\omega)+\int_0^t F(\omega,s,x(s))\,ds$ is measurable in ω, continuous in t. Hence

$$(\omega,t)\to d(x(t))_{x_0(\omega)+\int_0^t F(\omega,s,x(s))\,ds}$$

is Carathéodory. Now fix (ω,t). We claim that $x(\cdot)\to\phi(\omega,t,x(\cdot))$ is continuous. To see that let $x_n(\cdot)\to^{C_X} x(\cdot)$. Then we have

$$\begin{aligned}
&|d(x_n(t))_{x_0(\omega)+\int_0^t F(\omega,s,x_n(s))}-d(x(t))_{x_0(\omega)+\int_0^t F(\omega,s,x(s))}|\\
&\quad\leqslant |d(x_n(t))_{x_0(\omega)+\int_0^t F(\omega,s,x_n(s))}-d(x(t))_{x_0(\omega)+\int_0^t F(\omega,s,x_n(s))}|\\
&\qquad+|d(x(t))_{x_0(\omega)+\int_0^t F(\omega,s,x_n(s))}-d(x(t))_{x_0(\omega)+\int_0^t F(\omega,s,x(s))}|\\
&\quad\leqslant \|x_n(t)-x(t)\|+|d(x(t))_{x_0(\omega)+\int_0^t F(\omega,s,x_n(s))}-d(x(t))_{x_0(\omega)+\int_0^t F(\omega,s,x_n(s))}|.
\end{aligned}$$

Because $x_n(\cdot)\to^{C_X}x(\cdot)$ and $F(\omega,s,\cdot)$ is K–M-continuous then

$$\begin{aligned}
&F(\omega,s,x_n(s))\xrightarrow{\text{K M}}F(\omega,s,x(s))\\
&\quad\Rightarrow\int_0^t F(\omega,s,x_n(s))\,ds\xrightarrow{\text{K-M}}\int_0^t F(\omega,s,x(s))\,ds\\
&\quad\Rightarrow d(x(t))_{x_0(\omega)+\int_0^t F(\omega,s,x_n(s))\,ds}\to d(x(t))_{x_0(\omega)+\int_0^t F(\omega,s,x(s))\,ds}\quad\text{as } n\to\infty.
\end{aligned}$$

Thus by passing, to the limit as $n\to\infty$ we finally have that

$$\begin{aligned}
&d(x_n(t))_{x_0(\omega)+\int_0^t F(\omega,s,x_n(s))}\to d(x(t))_{x_0(\omega)+\int_0^t F(\omega,s,x(s))}\\
&\quad\Rightarrow\phi(\omega,t,x_n(\cdot))\to\phi(\omega,t,x(\cdot))
\end{aligned}$$

which shows that $\phi(\omega,t,\cdot)$ is continuous on $C_X(T)$. Recalling that $C_X(T)$ is a separable metric space, we deduce that $(\omega,x(\cdot))\to\phi(\omega,t,x(\cdot))$ is measurable. Let $\{t_n\}_{n\geqslant 1}$ be a dense set in T. Then $u(\omega,x(\cdot))=\sup_{n\geqslant 1}\phi(\omega,t_n,x(\cdot))$ is measurable. Now observe that

$$\begin{aligned}
&R(\omega)=\{x(\cdot)\in C_X(T)\colon u(\omega,x(\cdot))=0\}\\
&\Rightarrow \operatorname{Gr} R=\{(\omega,x(\cdot))\in\Omega\times C_X(T)\colon u(\omega,x(\cdot))=0\}\in\Sigma\times B(C_X(T)).
\end{aligned}$$

Apply Aumann's selection theorem to find $r: \Omega \to C_X(T)$ measurable s.t. $r(\omega) \in R(\omega)$ for all $\omega \in \Omega$. Set $r(\omega)(\cdot) = x(\omega, \cdot)$. Then $x(\cdot, \cdot)$ is a stochastic process with continuous paths and for all $(\omega, t) \in \Omega \times T$ we have $x(\omega, t) \in x_0(\omega) + \int_0^t F(\omega, s, x(\omega, s))\, ds \Rightarrow \dot{x}(\omega, t) \in F(\omega, t, x(\omega, t))$ a.e. for all $\omega \in \Omega$ and $x(\omega, 0) = x_0(\omega)$, i.e., $x(\cdot, \cdot)$ solves (SP). Q.E.D.

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